TWO-PARAMETER METHOD FOR DESCRIBING THE NONLINEAR EVOLUTION OF NARROW-BAND WAVE TRAINS

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We consider the evolution of narrow-band wave trains of finite amplitude in a nonlinear dispersive system which is described by the Klein–Gordon equation with arbitrary polynomial nonlinearity. We use a new perturbative technique which allows the original wave equation to be reduced to a model equation for the wave train envelope (high-order nonlinear Schrödinger equation). The time derivative is expanded into an asymptotic series in two independent parameters which characterize the smallness of amplitudes (ε) and the slowness of their spatial variations (μ). In contrast to other perturbative methods in which these two parameters are taken equal (e.g., the multiple scale method), the two-parameter method produces no secular terms. The results of this study can be applied to investigating the propagation of ultrashort (femtosecond) pulses in optical fibers, to studying the wave events on a fluid surface, and to describing the Langmuir waves in hot plasmas.

1. Introduction

One of the remarkable properties of nonlinear dispersive systems is the possibility of the existence of steady progressive waves with finite amplitude due to the balance between dispersion and nonlinearity effects. In the absence of dissipation, the dispersion relation (dependence of wave frequency on wavelength) can be written as

$$\omega = \omega(k,\ |A|^2), \tag{1}$$

where $k \equiv 2\pi/\lambda$ is the wave number, ω is the wave frequency, and λ is the wavelength. In systems with no dispersion, the dependence on k is linear, i.e., $\omega = ck$. The constant c is called the wave phase speed. In the case of nonzero dispersion, the phase speed depends on wave number ($\partial^2\omega/\partial k^2 \neq 0$), so that wave trains spread out in space. In nonlinear systems, this dispersive spreading can be compensated by nonlinear effects, which manifest themselves in the dependence of wave frequency on wave amplitude A in dispersion relation (1).

For narrow-band wave trains, when $\Delta k \ll k$, the dispersion relation can be expanded into a Taylor series about the carrier wave number k_0 and frequency ω_0:

$$\omega - \omega_0 \equiv \Delta\omega = \left(\frac{\partial\omega}{\partial k}\right)_{k=k_0}\Delta k + \frac{1}{2}\left(\frac{\partial^2\omega}{\partial k^2}\right)_{k=k_0}\Delta k^2 + + \left(\frac{\partial\omega}{\partial|A|^2}\right)_{|A|=0}|A|^2 + \dots\ . \tag{2}$$

Going over to the operator equation for the amplitude A by the substitution $\Delta\omega \rightleftarrows \mathtt{i}\partial/\partial\tau$, $\Delta k \rightleftarrows -\mathtt{i}\partial/\partial\xi$ and omitting the high-order terms, we get

$$\mathtt{i}(A_\tau + a_1 A_\xi) - a_2 A_{\xi\xi} + a_{0,\,0,\,0}A|A|^2 = 0, \tag{3}$$

where $a_1 = (\partial\omega/\partial k)_{k=k_0}$, $a_2 = \frac{1}{2}(\partial^2\omega/\partial k^2)_{k=k_0}$, $a_{0,\,0,\,0} = (\partial\omega/\partial|A|^2)_{|A|=0}$, and the variables τ and ξ stand for some conventional time and coordinate with respect to which the amplitude of wave train envelope exhibits slow variations.

Equation (3) is called the nonlinear Schrödinger equation (NLSE). It is met, in particular, in various problems of nonlinear optics, plasma physics, and hydrodynamics, and it admits solutions in the form of solitons [1, Sec. 8]. NLSE takes into account the second-order dispersion effects (term $A_{\xi\xi}$) and the phase self-modulation (term $A|A|^2$). These terms are sufficient to describe the propagation of picosecond pulses in optical fibers [2]. Since solitons are formed by the balance of dispersion and nonlinearity, a lesser pulse width can be obtained when the carrier wavelength is chosen such that the coefficient a_2 of the dispersion term vanishes. So, high-order dispersive and nonlinear effects come to the forefront in femtosecond-pulse problems. These effects are described by generalized NLSEs (high-order NLSEs) [2, 3]. In the case of hydrodynamic wave propagation along a fluid surface, high-order NLSEs were derived, in particular, in [4–7]. Another distinctive feature of wave trains described by NLSE is their instability with respect to long-wave modulations (modulational instability) at $a_2\, a_{0,\,0,\,0} < 0$ [8, p. 640]. To determine the modulational instability conditions at $a_{0,\,0,\,0} = 0$, high-order terms are to be taken into account in Eq. (3).

There are several methods which allow the original equations of one or other physical process to be reduced to model evolution equations of NLSE type. They include the multiple scale method [1, 6, 9], Hamiltonian formalism (Zakharov method) [10–12], variational method [13], and reductive perturbation methods [14, 15]. In the multiple scale method, an unknown function $u(x, t)$ of coordinate and time is looked for in the form of asymptotic expansion in powers of a small nonlinearity parameter ϵ:

$$u(x,\, t) = \sum_{n=1}^{\infty} \epsilon^n u^{(n)}(x,\, t). \tag{4}$$

The wave motion is classified into slow one and fast one by introducing different time scales and different spatial scales:

$$T_n \equiv \mu^n t, \quad X_n \equiv \mu^n x.$$

The derivatives with respect to time and coordinate are expanded into the following series:

$$\frac{\partial}{\partial t} = \sum_{n=0}^{\infty} \mu^n \frac{\partial}{\partial T_n}, \qquad \frac{\partial}{\partial x} = \sum_{n=0}^{\infty} \mu^n \frac{\partial}{\partial X_n}, \tag{5}$$

the times T_n and coordinates X_n being assumed to be independent variables. A principal drawback of this method lies in the fact that the parameters ϵ and μ with different physical meanings (the former characterizing the smallness of nonlinearity, and the latter describing the slowness of temporal and spatial variations) are tentatively taken equal: $\epsilon = \mu$. This admission produces the so-called secular terms in the equations for $u^{(n)}(x, t)$. Such terms, which infinitely grow with time, are eliminated in each new order of ϵ by an appropriate choice of free parameters emerging in the solutions of the linear inhomogeneous wave equations derived from the original nonlinear equations for the function $u(x, t)$. The procedure is very awkward, and it is difficult to formulate in algorithmic form. In Zakharov's method, the problem is reduced to an integral equation in the Fourier space, and the corresponding solutions should be transformed back to the physical space with the use of the inverse Fourier transformation. Again, the procedure is quite laborious. The same remarks can be made regarding all other methods mentioned above.

In work [16], V.P. Lukomsky proposed an idea of constructing a perturbation procedure free of secular terms. It was used to derive a generalized NLSE for the modulations of gravity waves on deep water. The method allows the original system of nonlinear equations to be reduced to a model equation for the pulse envelope in the form of asymptotic expansion of the time derivative in terms of two independent parameters which characterize the smallness of amplitudes (ε) and the slowness of their spatial variations (μ). In this paper, we present a general realization of this two-parameter procedure by the example of the reduction of the Klein–Gordon equation to a generalized NLSE. Our technique allows the coefficients of the generalized NLSE to be calculated in arbitrary order of ε (high-order nonlinear terms) and μ (high-order dispersive terms) as well as any their combination (nonlinear-dispersive terms).

Consider some wave process described by the $(1+1)$ Klein–Gordon equation with arbitrary polynomial nonlinearity:

$$u_{tt} - c^2 u_{xx} + \sum_{p=1}^{P} \alpha_p u^p = 0. \tag{6}$$

Here u is an unknown twice differentiable function of the wave process, $0 < t < \infty$ is time, $-\infty < x < \infty$ is coordinate, c and α_p are arbitrary real constants ($\alpha_1 \neq 0$), and P is an arbitrary positive integer. Let the initial condition at $t = 0$ have the form $u(x, 0) = Q(x)\big(\exp(\mathtt{i}kx) + \exp(-\mathtt{i}kx)\big)$, $u_t(x, 0) = P(x)\big(\exp(\mathtt{i}kx) + \exp(-\mathtt{i}kx)\big)$, where k is the carrier wave number.

The Klein–Gordon equation arises in the field theory, elementary particle physics, crystal dislocation models, etc. [1]. When $\alpha_{2p+1} = (-1)^p/(2p+1)!$, $\alpha_{2p} = 0$, $P = \infty$, Eq. (6) is called the sin-Gordon equation, and it is used to model the dynamics of dislocations in crystals, self-induced transparency in nonlinear optics, spin waves in fluid helium, propagation of fluxons in long Josephson (superconductive) junctions, and dynamics of domain walls in ferromagnetics [8, p. 840].

2. Spectral Representation

We look for a solution to Eq. (6) in the form of truncated Fourier series with variable coefficients:

$$u(x,\, t) = \sum_{n=-N_u}^{N_u} u_n(x,\, t)\mathrm{e}^{\mathtt{i}n(\omega t - kx)}, \quad u_{-n} \equiv u_n^*, \tag{7}$$

where ω is the wave-train carrier frequency, $N_u + 1$ is the number of harmonics taken into consideration, and * stands for complex conjugate. The same series can be written for all integer powers of the function u:

$$u^p(x,\, t) = \sum_{n=-pN_u}^{pN_u} (u^p)_n(x,\, t)\mathrm{e}^{\mathtt{i}n(\omega t - kx)}, \quad p = \overline{2,\, P}, \tag{8}$$

where $(u^p)_{-n} \equiv (u^p)^*_n$. The coefficients $(u^p)_n$ can be expressed recurrently in terms of the coefficients u_n [17, p. 30]:

$$(u^p)_n = \sum_{n_1=\max(-N_u,\, n-(p-1)N_u)}^{\min(N_u,\, n+(p-1)N_u)} u_{n_1}(u^{p-1})_{n-n_1}.$$

The corresponding expansions of the derivatives are

$$u_{tt}(x,\, t) = \sum_{n=-N_u}^{N_u} \big((u_n)_{tt} + 2\mathtt{i}n\omega(u_n)_t - \\ - n^2\omega^2 u_n\big)\mathrm{e}^{\mathtt{i}n(\omega t - kx)}, \tag{9}$$

$$u_{xx}(x,\, t) = \sum_{n=-N_u}^{N_u} \big((u_n)_{xx} - 2\mathtt{i}nk(u_n)_x - \\ - n^2k^2 u_n\big)\mathrm{e}^{\mathtt{i}n(\omega t - kx)}. \tag{10}$$

Substituting (7)–(10) in (6) and equating the coefficients at the like powers of the exponent $\exp(\mathtt{i}(\omega t - kx))$, we obtain a system of nonlinear differential equations for the coefficients $u_n(x,\, t)$ $(n = \overline{0,\, N_u})$:

$$(u_n)_{tt} - c^2(u_n)_{xx} + 2\mathtt{i}n\big(\omega(u_n)_t + c^2k(u_n)_x\big) + \\ +(n^2c^2k^2 - n^2\omega^2 + \alpha_1)u_n + \sum_{p=2}^{P}\alpha_p(u^p)_n = 0. \tag{11}$$

Linearization of these equations at $n = 1$ gives the dispersion relation in the linear approximation:

$$\omega^2 = \alpha_1 + c^2k^2. \tag{12}$$

3. Two-Parameter Expansions for Narrow-Band Wave Trains

Generally, the system of equations (11) is by no means more simple than original equation (6). It can be simplified if solutions are looked for in a class of functions with narrow spectrum, $|\Delta k| \ll k$ (*quasi-monochromaticity* condition). In this case, the problem has a formal small parameter $\mu \sim |\Delta k|/k$, and the coefficients $u_n(x,\, t)$ can be regarded as slow functions of x and t. Let us introduce a slow coordinate $\xi = \mu x$ and go over to the variables $u_n = u_n(\mu x,\, t)$.

When there are no resonances between higher harmonics, the amplitudes of Fourier coefficients decrease with increasing number (*quasi-harmonicity* condition):

$$u_n \sim \varepsilon^n A,\;\; n \geqslant 1, \quad u_0 \sim \varepsilon^2 A, \quad \varepsilon < 1, \tag{13}$$

where $u_1 \equiv \varepsilon A$. The parameter ε can be chosen as a second formal parameter, which is independent of the dispersion parameter μ in the general case. The use of two independent formal parameters is a distinctive feature of our approach as compared to other perturbative methods (e.g., multiple scale method), where these parameters are not distinguished ($\varepsilon = \mu$). When these incomparable parameters are set equal, a perturbative procedure produces non-physical secular terms.

In contrast to perturbative methods which use the expansions of form (4) and (5) to reduce Eqs. (11) to evolution equations of NLSE type (3), we immediately start from the most general explicit form of such an evolution equation. To this end, the time derivative $(u_1)_t \equiv \varepsilon A_t$ should be expressed in terms of the derivatives $(u_1)_{nx} \equiv \varepsilon\mu^n A_{n\xi}$ with respect to coordinate (designation $A_{n\xi}$ means the n-th derivative with respect to ξ) and all possible combinations of nonlinear terms $\varepsilon^{2n+1}A^{(n+1)}(A^*)^n$. Hence, the derivative A_t can be written as the following asymptotic expansion in terms of parameters ε and μ:

$$A_t = \mathtt{i}\sum_{n_0=0}^{\infty}(\mathtt{i}\mu)^{n_0}\Big(a_{n_0}A_{n_0\xi} + \\ +\varepsilon^2\sum_{n_1=0}^{n_0}\sum_{n_2=0}^{n_1} a_{n_0-n_1,\, n_1-n_2,\, n_2}A_{(n_0-n_1)\xi}\times \\ \times A_{(n_1-n_2)\xi}A^*_{n_2\xi} + O(\varepsilon^4)\Big). \tag{14}$$

Expression (14) is the general form of the evolution equation for the complex amplitude A of the first harmonic. The unknown coefficients $a_{n\_}$ can be determined from Eqs. (11). To this end, the amplitudes of all other harmonics ($u_0,\, u_2,\, u_3,\, \ldots$) are expanded in terms of the amplitude of the first harmonic A in the same manner as it is done in expansion (14):

$$u_0 = \varepsilon^2\sum_{n_0=0}^{\infty}(\mathtt{i}\mu)^{n_0}\Big(\sum_{n_1=0}^{n_0} b^{(0)}_{n_0-n_1,\, n_1}A_{(n_0-n_1)\xi}A^*_{n_1\xi} + \\ +\varepsilon^2\sum_{n_1=0}^{n_0}\sum_{n_2=0}^{n_1}\sum_{n_3=0}^{n_2} b^{(0)}_{n_0-n_1,\, n_1-n_2,\, n_2-n_3,\, n_3}\times \\ \times A_{(n_0-n_1)\xi}A_{(n_1-n_2)\xi}A^*_{(n_2-n_3)\xi}A^*_{n_3\xi} + O(\varepsilon^4)\Big), \tag{15}$$

$$u_2 = \varepsilon^2\sum_{n_0=0}^{\infty}(\mathtt{i}\mu)^{n_0}\Big(\sum_{n_1=0}^{n_0} b^{(2)}_{n_0-n_1,\, n_1}A_{(n_0-n_1)\xi}A_{n_1\xi} +$$

$$+\varepsilon^2 \sum_{n_1=0}^{n_0} \sum_{n_2=0}^{n_1} \sum_{n_3=0}^{n_2} b^{(2)}_{n_0-n_1,\, n_1-n_2,\, n_2-n_3,\, n_3} \times$$

$$\times A_{(n_0-n_1)\xi} A_{(n_1-n_2)\xi} A_{(n_2-n_3)\xi} A^*_{n_3\xi} + O(\varepsilon^4)\Big), \qquad (16)$$

$$u_3 = \varepsilon^3 \sum_{n_0=0}^{\infty} (\mathtt{i}\mu)^{n_0} \Big(\sum_{n_1=0}^{n_0} \sum_{n_2=0}^{n_1} b^{(3)}_{n_0-n_1,\, n_1-n_2,\, n_2} \times$$

$$\times A_{(n_0-n_1)\xi} A_{(n_1-n_2)\xi} A_{n_2\xi} +$$

$$+\varepsilon^2 \sum_{n_1=0}^{n_0} \sum_{n_2=0}^{n_1} \sum_{n_3=0}^{n_2} \sum_{n_4=0}^{n_3} b^{(3)}_{n_0-n_1,\, n_1-n_2,\, n_2-n_3\, n_3-n_4,\, n_4} \times$$

$$\times A_{(n_0-n_1)\xi} A_{(n_1-n_2)\xi} A_{(n_2-n_3)\xi} A_{(n_3-n_4)\xi} A^*_{n_4\xi} + O(\varepsilon^4)\Big), \qquad (17)$$

$\cdots$.

The unknown coefficients $b^{(n)}_{n\_}$ are found along with the coefficients $a_{n\_}$ from the system of equations (11) by substituting expressions (14)–(17) and equating the coefficients at the like powers of the products $\varepsilon^k \mu^m$ in different combinations $(A\_ \ldots A^*\_ \ldots)$ to zero. In its essence, this procedure is similar to the method of undetermined coefficients. The coefficient calculation order and the general form of the expansions for A_t and u_n in arbitrary order of ε are given in Appendix. The use of two parameters in ansatz (14) is of key importance for the coefficient calculation procedure, since the expansions could not be split into linear-independent terms at $\varepsilon = \mu$.

Note that our two-parameter approach has the same limitations in regard to convergence issues as other perturbative methods do. The convergence can get broken in the presence of resonances between harmonics, when quasi-harmonicity condition (13) is violated. Some questions related to the convergence of asymptotic expansions for the solutions of differential equations were considered, in particular, in our works [18,19]. Expansion (14) cannot be used either for wide-band wave trains with $\Delta k \sim k$.

It should also be noted that the reduction of Eq. (6) with the second time derivative to Eq. (14) with the first time derivative puts a constraint on the initial condition for u_t. In this case, $u_t(x,0)$ is a function of $u(x,0)$ defined by formula (14).

4. High-Order Nonlinear Schrödinger Equation

The two-parameter expansions were programmed in symbolic form for an arbitrary order of μ and ε. The evolution equation for the complex amplitude of the first harmonic is

$$A_t = \mathtt{i}\Big((\mathtt{i}\mu) a_1 A_\xi + (\mathtt{i}\mu)^2 a_2 A_{\xi\xi} +$$

$$+(\mathtt{i}\mu)^3 a_3 A_{\xi\xi\xi} + (\mathtt{i}\mu)^4 a_4 A_{\xi\xi\xi\xi} + O(\mu^5) +$$

$$+\varepsilon^2 \Big[a_{0,0,0} A|A|^2 + (\mathtt{i}\mu)\ \big(a_{1,0,0} A_\xi |A|^2 + a_{0,0,1} A^2 A^*_\xi\big) +$$

$$+(\mathtt{i}\mu)^2 \big(a_{2,0,0} A_{\xi\xi}|A|^2 + a_{1,1,0} A^2_\xi A^* +$$

$$+a_{1,0,1}|A_\xi|^2 A + a_{0,0,2} A^2 A^*_{\xi\xi}\big) + O(\mu^3)\Big] +$$

$$+\varepsilon^4 \Big[a_{0,0,0,0,0} A|A|^4 +$$

$$+(\mathtt{i}\mu)\ \big(a_{1,0,0,0,0} A_\xi |A|^4 + a_{0,0,0,1,0} A^2 |A|^2 A^*_\xi\big) +$$

$$+(\mathtt{i}\mu)^2 \big(a_{2,0,0,0,0} A_{\xi\xi} |A|^4 + a_{1,1,0,0,0} A^2_\xi |A|^2 A^* +$$

$$+a_{1,0,0,1,0} |A_\xi|^2 A |A|^2 + a_{0,0,0,2,0} A^2 |A|^2 A^*_{\xi\xi} +$$

$$+a_{0,0,0,1,1} A^3 A^2_\xi\big) + O(\mu^3)\Big] + O(\varepsilon^6)\Big). \qquad (18)$$

In each term of this equation, the power of the formal parameter μ points to the overall order of the derivatives with respect to ξ, and the power of the formal parameter ε points to the nonlinearity order. These parameters disappear after going back to the original variables $u_1 = \varepsilon A$ and $x = \xi/\mu$. Taking into account dispersion relation (12), the coefficients $a\_$ can be written as (at $P = 5$)

$$a_0 = 0, \quad a_1 = \frac{c^2 k}{\omega}, \quad a_2 = \frac{c^2 \alpha_1}{2\omega^3}, \quad a_3 = -\frac{c^4 k \alpha_1}{2\omega^5},$$

$$a_4 = \frac{\alpha_1 c^4 (4c^2k^2 - \alpha_1)}{8\omega^7}, \quad a_n = \frac{1}{n!} \frac{\mathrm{d}^n \omega}{\mathrm{d}k^n};$$

$$a_{0,0,0} = \frac{3\alpha_3}{2\omega} - \frac{5\alpha_2^2}{3\omega\alpha_1},$$

$$a_{1,0,0} = 2a_{0,0,1} = \frac{c^2 k}{\omega^3}\Big(\frac{10\alpha_2^2}{3\alpha_1} - 3\alpha_3\Big);$$

$$a_{2,0,0} = \frac{c^2}{18\alpha_1\omega^5}\Big(2c^2k^2\big(27\alpha_1\alpha_3 - 14\alpha_2^2\big) -$$

$$-\alpha_1\big(27\alpha_1\alpha_3 - 62\alpha_2^2\big)\Big),$$

$$a_{1,1,0} = \frac{c^2}{36\alpha_1\omega^5}\Big(4c^2k^2\big(27\alpha_1\alpha_3 - 28\alpha_2^2\big) -$$

$$-\alpha_1\big(27\alpha_1\alpha_3 - 38\alpha_2^2\big)\Big),$$

$$a_{1,0,1} = \frac{c^2}{6\alpha_1\omega^5}\Big(4c^2k^2\big(9\alpha_1\alpha_3 - 4\alpha_2^2\big) -$$

$$-\alpha_1\big(9\alpha_1\alpha_3 - 34\alpha_2^2\big)\Big),$$

$$a_{0,\,0,\,2}=\frac{c^2}{6\alpha_1\omega^5}\Big(c^2k^2\big(9\alpha_1\alpha_3+2\alpha_2^2\big)+12\alpha_1\alpha_2^2\Big);$$

$$a_{0,\,0,\,0,\,0,\,0}=\frac{1}{\omega}\Big(-\frac{335\alpha_2^4}{108\alpha_1^3}-\frac{25\alpha_2^4}{18\omega^2\alpha_1^2}+\frac{143\alpha_2^2\alpha_3}{12\alpha_1^2}+$$

$$+\frac{5\alpha_2^2\alpha_3}{2\omega^2\alpha_1}-\frac{9\alpha_3^2}{8\omega^2}+\frac{3\alpha_3^2}{16\alpha_1}-\frac{14\alpha_2\alpha_4}{\alpha_1}+5\alpha_5\Big),$$

$$a_{1,\,0,\,0,\,0,\,0}=\frac{c^2k}{\omega^3}\Big(\frac{925\alpha_2^4}{108\alpha_1^3}+\frac{275\alpha_2^4}{18\omega^2\alpha_1^2}-\frac{421\alpha_2^2\alpha_3}{12\alpha_1^2}-$$

$$-\frac{55\alpha_2^2\alpha_3}{2\omega^2\alpha_1}+\frac{99\alpha_3^2}{8\omega^2}-\frac{9\alpha_3^2}{16\alpha_1}+\frac{42\alpha_2\alpha_4}{\alpha_1}-15\alpha_5\Big),$$

$$a_{0,\,0,\,0,\,1,\,0}=\frac{c^2k}{\omega^3}\Big(\frac{295\alpha_2^4}{54\alpha_1^3}+\frac{100\alpha_2^4}{9\omega^2\alpha_1^2}-\frac{139\alpha_2^2\alpha_3}{6\alpha_1^2}-$$

$$-\frac{20\alpha_2^2\alpha_3}{\omega^2\alpha_1}+\frac{9\alpha_3^2}{\omega^2}-\frac{3\alpha_3^2}{8\alpha_1}+\frac{28\alpha_2\alpha_4}{\alpha_1}-10\alpha_5\Big).$$

The expressions for the subsequent coefficients $a\_$ are too long to be presented in explicit form.

The complex amplitudes of other harmonics are found from the relations

$$u_0=\varepsilon^2\Big(\Big[b^{(0)}_{0,\,0}|A|^2+(\mathtt{i}\mu)\big(b^{(0)}_{1,\,0}A_\xi A^*+b^{(0)}_{0,\,1}AA^*_\xi\big)+$$

$$+(\mathtt{i}\mu)^2\big(b^{(0)}_{2,\,0}A_{\xi\xi}A^*+b^{(0)}_{1,\,1}A_\xi A^*_\xi+b^{(0)}_{0,\,2}AA^*_{\xi\xi}\big)+$$

$$+O(\mu^3)\Big]+\varepsilon^2\Big[b^{(0)}_{0,\,0,\,0,\,0}|A|^4+$$

$$+(\mathtt{i}\mu)\ \big(b^{(0)}_{1,\,0,\,0,\,0}A_\xi|A|^2A^*+b^{(0)}_{0,\,0,\,1,\,0}A|A|^2A^*_\xi\big)+$$

$$+(\mathtt{i}\mu)^2\big(b^{(0)}_{2,\,0,\,0,\,0}A_{\xi\xi}|A|^2A^*+b^{(0)}_{1,\,1,\,0,\,0}A_\xi^2A^{*2}+$$

$$+b^{(0)}_{1,\,0,\,1,\,0}|A_\xi|^2|A|^2+b^{(0)}_{0,\,0,\,2,\,0}A|A|^2A^*_{\xi\xi}+$$

$$+b^{(0)}_{0,\,0,\,1,\,1}A^2A^{*2}_\xi\big)+O(\mu^3)\Big]+O(\varepsilon^4)\Big);$$

$$u_2=\varepsilon^2\Big(\Big[b^{(2)}_{0,\,0}A^2+(\mathtt{i}\mu)b^{(2)}_{1,\,0}A_\xi A+$$

$$+(\mathtt{i}\mu)^2\big(b^{(2)}_{2,\,0}A_{\xi\xi}A+b^{(2)}_{1,\,1}A_\xi^2\big)+O(\mu^3)\Big]+$$

$$+\varepsilon^2\Big[b^{(2)}_{0,\,0,\,0,\,0}A^2|A|^2+$$

$$+(\mathtt{i}\mu)\ \big(b^{(2)}_{1,\,0,\,0,\,0}A_\xi A|A|^2+b^{(2)}_{0,\,0,\,0,\,1}A^3A^*_\xi\big)+$$

$$+(\mathtt{i}\mu)^2\big(b^{(2)}_{2,\,0,\,0,\,0}A_{\xi\xi}A|A|^2+b^{(2)}_{1,\,1,\,0,\,0}A_\xi^2|A|^2+$$

$$+b^{(2)}_{1,\,0,\,0,\,1}|A_\xi|^2A^2+b^{(2)}_{0,\,0,\,0,\,2}A^3A^*_{\xi\xi}\big)+O(\mu^3)\Big]+O(\varepsilon^4)\Big);$$

$$u_3=\varepsilon^3\Big(\Big[b^{(3)}_{0,\,0,\,0}A^3+(\mathtt{i}\mu)b^{(3)}_{1,\,0,\,0}A_\xi A^2+$$

$$+(\mathtt{i}\mu)^2\big(b^{(3)}_{2,\,0,\,0}A_{\xi\xi}A^2+b^{(3)}_{1,\,1,\,0}A_\xi^2A\big)+O(\mu^3)\Big]+O(\varepsilon^2)\Big).$$

At $P=5$, the coefficients of the expansions are

$$b^{(0)}_{0,\,0}=-\frac{2\alpha_2}{\alpha_1};\quad b^{(0)}_{1,\,0}=b^{(0)}_{0,\,1}=0;$$

$$b^{(0)}_{2,\,0}=\frac{1}{2}\,b^{(0)}_{1,\,1}=b^{(0)}_{0,\,2}=\frac{2c^2\alpha_2}{\omega^2\alpha_1};$$

$$b^{(0)}_{0,\,0,\,0,\,0}=-\frac{38\alpha_2^3}{9\alpha_1^3}+\frac{10\alpha_2\alpha_3}{\alpha_1^2}-\frac{6\alpha_4}{\alpha_1};$$

$$b^{(0)}_{1,\,0,\,0,\,0}=b^{(0)}_{0,\,0,\,1,\,0}=0;$$

$$b^{(0)}_{2,\,0,\,0,\,0}=b^{(0)}_{0,\,0,\,2,\,0}=$$

$$=\frac{c^2}{27\alpha_1^3\omega^4}\Big(-4c^2k^2\left(23\alpha_2^3+90\alpha_1\alpha_2\alpha_3-81\alpha_1^2\alpha_4\right)+$$

$$+\alpha_1\left(538\alpha_2^3-927\alpha_1\alpha_2\alpha_3+324\alpha_1^2\alpha_4\right)\Big),$$

$$b^{(0)}_{1,\,1,\,0,\,0}=b^{(0)}_{0,\,0,\,1,\,1}=$$

$$=\frac{c^2}{27\alpha_1^3\omega^4}\Big(-4c^2k^2\left(79\alpha_2^3+18\alpha_1\alpha_2\alpha_3-81\alpha_1^2\alpha_4\right)+$$

$$+\alpha_1\left(314\alpha_2^3-639\alpha_1\alpha_2\alpha_3+324\alpha_1^2\alpha_4\right)\Big),$$

$$b^{(0)}_{1,\,0,\,1,\,0}=$$

$$=\frac{4c^2}{9\alpha_1^3\omega^4}\Big(-4c^2k^2\left(17\alpha_2^3+18\alpha_1\alpha_2\alpha_3-27\alpha_1^2\alpha_4\right)+$$

$$+\alpha_1\left(142\alpha_2^3-261\alpha_1\alpha_2\alpha_3+108\alpha_1^2\alpha_4\right)\Big);$$

$$b^{(2)}_{0,\,0}=\frac{\alpha_2}{3\alpha_1};\quad b^{(2)}_{1,\,0}=0;\quad b^{(2)}_{2,\,0}=-b^{(2)}_{1,\,1}=-\frac{2c^2\alpha_2}{9\omega^2\alpha_1};$$

$$b^{(2)}_{0,\,0,\,0,\,0}=\frac{59\alpha_2^3}{54\alpha_1^3}-\frac{31\alpha_2\alpha_3}{12\alpha_1^2}+\frac{4\alpha_4}{3\alpha_1};$$

$$b^{(2)}_{1,\,0,\,0,\,0}=b^{(2)}_{0,\,0,\,0,\,1}=\frac{2c^2k\alpha_2}{27\omega^2\alpha_1^3}\big(9\alpha_1\alpha_3-10\alpha_2^2\big);$$

$$b^{(2)}_{2,\,0,\,0,\,0}=-\frac{c^2}{1296\alpha_1^3\omega^4}\Big(c^2k^2\big(2606\alpha_2^3-5283\alpha_1\alpha_2\alpha_3+$$

$$+1728\alpha_1^2\alpha_4\big)+\alpha_1\left(5006\alpha_2^3-7443\alpha_1\alpha_2\alpha_3+1728\alpha_1^2\alpha_4\right)\Big),$$

$$b^{(2)}_{1,\,1,\,0,\,0}=\frac{c^2}{1296\alpha_1^3\omega^4}\Big(c^2k^2\big(3574\alpha_2^3-7191\alpha_1\alpha_2\alpha_3+$$

$$+3456\alpha_1^2\alpha_4\big)+\alpha_1\left(3094\alpha_2^3-6759\alpha_1\alpha_2\alpha_3+3456\alpha_1^2\alpha_4\right)\Big),$$

$$b^{(2)}_{1,\,0,\,0,\,1}=\frac{c^2}{162\alpha_1^3\omega^4}\Big(c^2k^2\big(122\alpha_2^3-369\alpha_1\alpha_2\alpha_3+$$

$$+432\alpha_1^2\alpha_4\big)+\alpha_1\left(-358\alpha_2^3+63\alpha_1\alpha_2\alpha_3+432\alpha_1^2\alpha_4\right)\Big),$$

$$b^{(2)}_{0,\,0,\,0,\,2} = \frac{c^2}{324\alpha_1^3\omega^4}\Big(c^2k^2\big(2\alpha_2^3 - 261\alpha_1\alpha_2\alpha_3 +$$

$$+432\alpha_1^2\alpha_4\big) + \alpha_1\left(-238\alpha_2^3 - 45\alpha_1\alpha_2\alpha_3 + 432\alpha_1^2\alpha_4\right)\Big);$$

$$b^{(3)}_{0,\,0,\,0} = \frac{\alpha_2^2}{12\alpha_1^2} + \frac{\alpha_3}{8\alpha_1}; \quad b^{(3)}_{1,\,0,\,0} = 0;$$

$$b^{(3)}_{2,\,0,\,0} = -b^{(3)}_{1,\,1,\,0} = -\frac{c^2}{288\alpha_1^2\omega^2}\left(27\alpha_1\alpha_3 + 34\alpha_2^2\right).$$

Some of these coefficients were derived earlier in [9] by the multiple scale method. The expressions presented in [9] are in agreement with those obtained here (except for several misprints and typographic errors).

5. The Effect of High-Order Dispersive Terms

Let us illustrate the evolution of a wave train envelope described by the equation of form (14). To this end, we rewrite original equation (6) in dimensionless variables $\widetilde{x} \equiv kx$ and $\widetilde{t} \equiv ckt$:

$$\widetilde{u}_{\widetilde{t}\widetilde{t}} - \widetilde{u}_{\widetilde{x}\widetilde{x}} + \sum_{p=1}^{P}\widetilde{\alpha}_p\widetilde{u}^p = 0, \quad \widetilde{u} = \frac{u}{U_0}, \quad \widetilde{\alpha}_p = \frac{\alpha_p U_0^{p-1}}{(ck)^2}. \tag{19}$$

In this case, we have $\widetilde{c} = 1$, $\widetilde{k} = 1$, $\widetilde{\omega}^2 = \widetilde{\alpha}_1 + 1$, and U_0 is a typical amplitude of the function u.

As an example, let us consider the case $P = 3$ with $\widetilde{\alpha}_1 = 1$, $\widetilde{\alpha}_2 = 0$, and $\widetilde{\alpha}_3 = -1/6$. The values of these parameters correspond to the first two terms in the Taylor expansion of the function $\sin u$. Hereafter, the tildes over the dimensionless variables are omitted.

The corresponding coefficients of evolution equation (18) are

$$a_1 = \frac{1}{\sqrt{2}}, \quad a_2 = \frac{1}{4\sqrt{2}}, \quad a_3 = -\frac{1}{8\sqrt{2}}, \quad a_4 = \frac{3}{64\sqrt{2}},$$

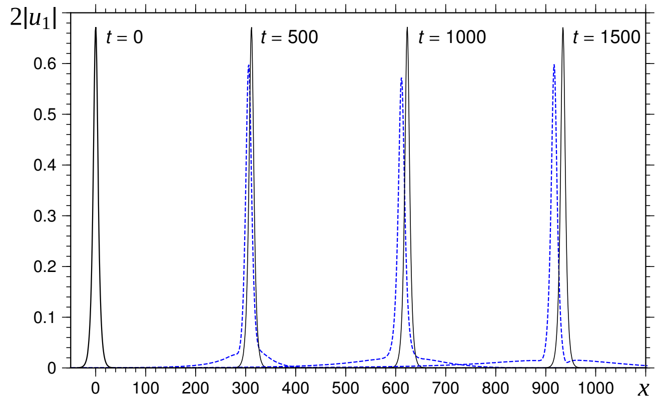
$$a_{0,\,0,\,0} = -\frac{1}{4\sqrt{2}}, \quad a_{1,\,0,\,0} = \frac{1}{4\sqrt{2}}, \quad a_{0,\,0,\,1} = \frac{1}{8\sqrt{2}},$$

$$a_{2,\,0,\,0} = a_{0,\,0,\,2} = -\frac{1}{16\sqrt{2}}, \quad a_{1,\,1,\,0} = a_{1,\,0,\,1} = -\frac{3}{16\sqrt{2}},$$

$$a_{0,\,0,\,0,\,0,\,0} = -\frac{1}{96\sqrt{2}}, \quad \ldots\,. \tag{20}$$

Initially, we retain only those terms in Eq. (18) whose overall order of smallness with respect to the parameters ε and μ is no more than two. In this case, we obtain a classical NLSE:

$$(u_1)_t = -a_1(u_1)_x - \mathtt{i}a_2(u_1)_{xx} + \mathtt{i}a_{0,\,0,\,0}\,u_1|u_1|^2. \tag{21}$$



Evolution of the wave train envelope which is given, at the initial moment $t = 0$, by function (22) with parameters $\beta = \zeta = 1/10$ and $\phi_0 = x_0 = 0$. (Solid curve) exact one-soliton solution of NLSE (21), (dashed curve) numerical solution of the generalized NLSE (24)

It has an exact one-soliton solution at $a_2\,a_{0,\,0,\,0} < 0$:

$$u_1(x,\,t) = \beta\left(\frac{2}{|a_{0,\,0,\,0}|}\right)^{1/2}\times$$

$$\times\exp\left(\mathtt{i}\left(\frac{\zeta}{\sqrt{|a_2|}}(x - a_1t) - s(\zeta^2 - \beta^2)t + \phi_0\right)\right)\times$$

$$\times\cosh^{-1}\left(\frac{\beta}{\sqrt{|a_2|}}(x - x_0 - a_1t) - 2s\beta\zeta t\right). \tag{22}$$

Here $s = \operatorname{sign}(a_{0,\,0,\,0})$ and β, ζ, ϕ_0, and x_0 are free parameters [20, 21]. The corresponding approximate solution of Eq. (19) is

$$u(x,\,t) = u_1(x,\,t)\exp\big(\mathtt{i}(\sqrt{2}\,t - x)\big) +$$

$$+u_1^*(x,\,t)\exp\big(-\mathtt{i}(\sqrt{2}\,t - x)\big). \tag{23}$$

To analyze the effect of high-order dispersive terms on the shape of one-soliton solution (22), we consider the generalized NLSE

$$(u_1)_t = -a_1(u_1)_x - \mathtt{i}a_2(u_1)_{xx} + a_3(u_1)_{xxx} +$$

$$+\mathtt{i}a_4(u_1)_{xxxx} + \mathtt{i}a_{0,\,0,\,0}\,u_1|u_1|^2 -$$

$$-a_{1,\,0,\,0}(u_1)_x|u_1|^2 - a_{0,\,0,\,1}\,u_1^2(u_1^*)_x -$$

$$-\mathtt{i}a_{2,\,0,\,0}(u_1)_{xx}|u_1|^2 - \mathtt{i}a_{1,\,1,\,0}(u_1)_x^2\,u_1^* -$$

$$-\mathtt{i}a_{1,\,0,\,1}|(u_1)_x|^2u_1 - \mathtt{i}a_{0,\,0,\,2}\,u_1^2(u_1^*)_{xx} +$$

$$+\mathtt{i}a_{0,\,0,\,0,\,0,\,0}\,u_1|u_1|^4 \tag{24}$$

with the coefficients defined by (20). The initial condition is chosen in the form of function (22) with $\beta = \zeta = 1/10$ and $\phi_0 = x_0 = 0$. The figure shows the evolution of such an envelope. To solve Eq. (24) numerically, we used the split-step Fourier method [22, 23]. High-order dispersive terms are seen to affect the amplitude, shape, and velocity of the soliton solution.

6. Conclusion

We described a general method for deriving the evolution equations for narrow-band wave trains in nonlinear media with dispersion. The procedure produces no secular terms and can easily be put in algorithmic form. By using the Klein–Gordon equation with arbitrary polynomial nonlinearity as an example, we derived a generalized NLSE whose coefficients can be calculated in any order with respect to nonlinearity and dispersion. The equation can be used to investigate the propagation of ultrashort pulses in optical fibers, to study wave events on a fluid surface, and to describe the Langmuir waves in hot plasmas.

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Postscript

The major part of this paper was written during the last months of Vasyl Petrovich Lukomsky's lifetime (January 14, 1942 – March 31, 2008). The above-described two-parameter method was developed by him as far back as at the end of the 1980s (some of his notes regarding NLSE were dated by 1986 and 1987). Regretfully, the results of those studies were published with delay and only in one paper of 1995 [16]. Vasyl Petrovich was attracted by various fields of physics, he had very broad physical horizons and often sacrificed the time for paper preparation in favor of something new. Still unpublished is the paper "Method of indefinite coefficients for derivation of evolution equations with higher terms for nonlinear waves" written together with Yu.G. Rapoport and submitted to Physica Scripta in 1999. They derived a high-order NLSE for describing the propagation of ion sound in non-isothermal plasmas. Vasyl Petrovich turned back to his method (which can appropriately be called the Lukomsky method) in 2005 with the aim to derive a generalized NLSE for the modulations of gravity waves on a fluid surface. This paper was scheduled as the first out of the whole series of papers devoted to high-order evolution equations. To our deep sorrow, these plans were ruined by the fatal malady and untimely death of Vasyl Petrovich. Only time will show whether the work which had started can be finished without its inspirer.

Vasyl Petrovich was a kind, sincere, calm, even-tempered, tolerant, open, and generous man, who was always ready to help. He was faithful to his life principles and convictions till the end. He had ingenious non-standard way of thinking and well-trained intuition. He was a man of word and justice. Blessed memory about Vasyl Petrovich Lukomsky will always abide in the hearts of those people who had the honor to be in fellowship and collaboration with him.

APPENDIX

The general forms of the expansions for A_t and u_n are

$$A_t = \mathtt{i} \sum_{n_0=0}^{\infty} (\mathtt{i}\mu)^{n_0} \sum_{k=0}^{\infty} \varepsilon^{2k} (u_1)_{n_0,\,2k+1}\,,$$

$$(u_1)_{n_0,\,2k+1} = \\ = \sum_{n_1=0}^{n_0} \sum_{n_2=0}^{n_1} \cdots \sum_{n_{2k}=0}^{n_{2k-1}} a_{n_0-n_1,\,n_1-n_2,\,\ldots,\,n_{2k-1}-n_{2k},\,n_{2k}} \times \\ \times \prod_{i=1}^{k+1} A_{(n_{i-1}-n_i)\xi} \prod_{i=k+2}^{2k+1} A^*_{(n_{i-1}-n_i)\xi}\,, \quad n_{2k+1} \equiv 0;$$

$$u_n = \varepsilon^n \sum_{n_0=0}^{\infty} (\mathtt{i}\mu)^{n_0} \sum_{k=0}^{\infty} \varepsilon^{2k} (u_n)_{n_0,\,2k+n}\,, \; n > 2,$$

$$(u_n)_{n_0,\,2k+n} = \sum_{n_1=0}^{n_0} \sum_{n_2=0}^{n_1} \cdots \\ \cdots \sum_{n_{2k+n-1}=0}^{n_{2k+n-2}} b^{(n)}_{n_0-n_1,\,n_1-n_2,\,\ldots,\,n_{2k+n-2}-n_{2k+n-1},\,n_{2k+n-1}} \times \\ \times \prod_{i=1}^{k+n} A_{(n_{i-1}-n_i)\xi} \prod_{i=k+n+1}^{2k+n} A^*_{(n_{i-1}-n_i)\xi}\,, \quad n_{2k+n} \equiv 0;$$

$$u_0 = \varepsilon^2 \sum_{n_0=0}^{\infty} (\mathtt{i}\mu)^{n_0} \sum_{k=0}^{\infty} \varepsilon^{2k} (u_0)_{n_0,\,2k+2}\,,$$

$$(u_0)_{n_0,\,2k+2} = \\ = \sum_{n_1=0}^{n_0} \sum_{n_2=0}^{n_1} \cdots \sum_{n_{2k+1}=0}^{n_{2k}} b^{(0)}_{n_0-n_1,\,n_1-n_2,\,\ldots,\,n_{2k}-n_{2k+1},\,n_{2k+1}} \times \\ \times \prod_{i=1}^{k+1} A_{(n_{i-1}-n_i)\xi} \prod_{i=k+2}^{2k+2} A^*_{(n_{i-1}-n_i)\xi}\,, \quad n_{2k+2} \equiv 0.$$

The sums above contain many identical summands (e.g., $a_{1,\,0,\,0} A_\xi A A^*$ and $a_{0,\,1,\,0} A A_\xi A^*$). The repetitions can be eliminated if one limits the summation orders using the rules $n_{i-2} - n_{i-1} \leqslant n_{i-1} - n_i \leqslant n_i - n_{i+1}$. The corresponding expansions are

$$(u_1)_{n_0,\,2k+1} = \sum_{n_1=0}^{n_0} \sum_{n_2=\max(0,\,2n_1-n_0)}^{n_1} \cdots$$

Coefficient calculation order

Iteration number	Number of Eqs. (11)	Coefficient
$k=0$ (ε^1)	$n=1$ (ε^1)	a_{n_0}
$k=1$ (ε^3)	$n=0$ (ε^2)	$b^{(0)}_{n_0-n_1,\,n_1}$
	$n=2$ (ε^2)	$b^{(2)}_{n_0-n_1,\,n_1}$
	$n=1$ (ε^3)	$a_{n_0-n_1,\,n_1-n_2,\,n_2}$
$k=2$ (ε^5)	$n=0$ (ε^4)	$b^{(0)}_{n_0-n_1,\,n_1-n_2,\,n_2-n_3,\,n_3}$
	$n=3$ (ε^3)	$b^{(3)}_{n_0-n_1,\,n_1-n_2,\,n_2}$
	$n=2$ (ε^4)	$b^{(2)}_{n_0-n_1,\,n_1-n_2,\,n_2-n_3,\,n_3}$
	$n=1$ (ε^5)	$a_{n_0-n_1,\,n_1-n_2,\,n_2-n_3,\,n_3-n_4,\,n_4}$
$k=3$ (ε^7)	$n=0$ (ε^6)	$b^{(0)}_{n_0-n_1,\,n_1-n_2,\,n_2-n_3,\,n_3-n_4,\,n_4-n_5,\,n_5}$
	$n=4$ (ε^4)	$b^{(4)}_{n_0-n_1,\,n_1-n_2,\,n_2-n_3,\,n_3}$
	$n=3$ (ε^5)	$b^{(3)}_{n_0-n_1,\,n_1-n_2,\,n_2-n_3,\,n_3-n_4,\,n_4}$
	$n=2$ (ε^6)	$b^{(2)}_{n_0-n_1,\,n_1-n_2,\,n_2-n_3,\,n_3-n_4,\,n_4-n_5,\,n_5}$
	$n=1$ (ε^7)	$a_{n_0-n_1,\,n_1-n_2,\,n_2-n_3,\,n_3-n_4,\,n_4-n_5,\,n_5-n_6,\,n_6}$

$$\ldots \sum_{n_i=\max(0,\,2n_{i-1}-n_{i-2})}^{n_{i-1}} \ldots \sum_{n_{k+1}=\max(0,\,2n_k-n_{k-1})}^{n_k}$$

$$\sum_{n_{k+2}=0}^{[\frac{k-1}{k}n_{k+1}]} \ldots \sum_{n_i=\max(0,\,2n_{i-1}-n_{i-2})}^{[\frac{2k+1-i}{2k+2-i}n_{i-1}]} \ldots$$

$$\ldots \sum_{n_{2k}=\max(0,\,2n_{2k-1}-n_{2k-2})}^{[\frac{1}{2}n_{2k-1}]} a_{n_0-n_1,\,n_1-n_2,\,\ldots,\,n_{2k-1}-n_{2k},\,n_{2k}} \times$$

$$\times \prod_{i=1}^{k+1} A_{(n_{i-1}-n_i)\xi} \prod_{i=k+2}^{2k+1} A^*_{(n_{i-1}-n_i)\xi}\,, \quad n_{2k+1}\equiv 0;$$

$$(u_{n>1})_{n_0,\,2k+n}|_{k=0} = \sum_{n_1=0}^{[\frac{n-1}{n}n_0]} \ldots \sum_{n_i=\max(0,\,2n_{i-1}-n_{i-2})}^{[\frac{n-i}{n-i+1}n_{i-1}]} \ldots$$

$$\ldots \sum_{n_{n-1}=\max(0,\,2n_{n-2}-n_{n-3})}^{[\frac{1}{2}n_{n-2}]} b_-^{(n)} \prod_{i=1}^{n} A_-;$$

$$(u_0)_{n_0,\,2k+2}|_{k>0} = \sum_{n_1=0}^{n_0} \sum_{n_2=\max(0,\,2n_1-n_0)}^{n_1} \ldots$$

$$\ldots \sum_{n_i=\max(0,\,2n_{i-1}-n_{i-2})}^{n_{i-1}} \ldots \sum_{n_{k+1}=\max(0,\,2n_k-n_{k-1})}^{n_k}$$

$$\sum_{n_{k+2}=0}^{[\frac{k}{k+1}n_{k+1}]} \ldots \sum_{n_i=\max(0,\,2n_{i-1}-n_{i-2})}^{[\frac{2k-i+2}{2k-i+3}n_{i-1}]} \ldots$$

$$\ldots \sum_{n_{2k+1}=\max(0,\,2n_{2k}-n_{2k-1})}^{[\frac{1}{2}n_{2k}]} b_-^{(0)} \prod_{i=1}^{k+1} A_- \prod_{i=k+2}^{2k+2} A^*_-;$$

$$(u_0)_{n_0,\,2k+2}|_{k=0} = \sum_{n_1=0}^{n_0} b^{(0)}_{n_0-n_1,\,n_1} A_{(n_0-n_1)\xi} A^*_{n_1\xi}.$$

$$(u_{n>1})_{n_0,\,2k+n}|_{k>0} = \sum_{n_1=0}^{n_0} \sum_{n_2=\max(0,\,2n_1-n_0)}^{n_1} \ldots$$

$$\ldots \sum_{n_i=\max(0,\,2n_{i-1}-n_{i-2})}^{n_{i-1}} \ldots \sum_{n_{k+n}=\max(0,\,2n_{k+n-1}-n_{k+n-2})}^{n_{k+n-1}}$$

$$\sum_{n_{k+n+1}=0}^{[\frac{k-1}{k}n_{k+n}]} \ldots \sum_{n_i=\max(0,\,2n_{i-1}-n_{i-2})}^{[\frac{2k+n-i}{2k+n-i+1}n_{i-1}]} \ldots$$

$$\ldots \sum_{n_{2k+n-1}=\max(0,\,2n_{2k+n-2}-n_{2k+n-3})}^{[\frac{1}{2}n_{2k+n-2}]} b_-^{(n)} \prod_{i=1}^{k+n} A_- \prod_{i=k+n+1}^{2k+n} A^*_-;$$

The table gives the order of calculation of the coefficients $b_-^{(n)}$ and a_-.

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ДВОПАРАМЕТРИЧНИЙ МЕТОД ДЛЯ ОПИСУ НЕЛІНІЙНОЇ ЕВОЛЮЦІЇ СПЕКТРАЛЬНО ВУЗЬКИХ ХВИЛЬОВИХ ПАКЕТІВ

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Р е з ю м е

Розглянуто часову еволюцію спектрально вузьких хвильових пакетів кінцевої амплітуди в нелінійній дисперсійній системі, описуваній рівнянням Клейна–Ґордона з довільною поліноміальною нелінійністю. Застосовано новий метод теорії збурень, який дозволяє звести вихідне хвильове рівняння до модельного рівняння для обвідної хвильового пакету (нелінійне рівняння Шредінґера вищого порядку). Побудовано асимптотичне розвинення часової похідної по двох незалежних параметрах, котрі характеризують малість амплітуд (ε) і повільність їх змін у просторі (μ). На відміну від інших методів теорії збурень (таких як метод багатьох масштабів), де ці два параметри не розрізняють, двопараметричний метод не приводить до появи вікових (секулярних) доданків. Результати роботи можуть бути застосовані до дослідження поширення ультракоротких (фемтосекундних) імпульсів в мережах волокнисто-оптичного зв'язку, вивчення хвильових явищ на поверхні рідини, опису ленгмюрівських хвиль в гарячій плазмі.